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ON WEAKLY CYCLIC Z SYMMETRIC SPACETIMES

U. C. DE[1], C. A. MANTICA[2,3,*], L. G. MOLINARI[4] and Y. J. SUH[5,†]

[1]Department of Pure Mathematics, University of Calcutta, 35, Ballygaunge Circular Road, Kolkata 700019, West Bengal, India
e-mail: uc_de@yahoo.com

[2]Physics Department, Universitá degli Studi di Milano, Via Celoria 16, 20133, Milano, Italy

[3]I. I. S. Lagrange, Via L., Modignani 65, 20161 Milano, Italy
e-mail: carloalberto.mantica@libero.it

[4]Physics Department, Universitá degli Studi di Milano and I. N. F. N. Sez. Milano, Via Celoria 16, 20133, Milano, Italy
e-mail: luca.molinari@mi.infn.it

[5]Kyungpook National University, Department of Mathematics, Taegu 702-701, Korea
e-mail: yjsuh@knu.ac.kr

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Abstract. The object of the present paper is to study weakly cyclic Z symmetric spacetimes. At first we prove that a weakly cyclic Z symmetric spacetime is a quasi Einstein spacetime. Then we study $(WCZS)_4$ spacetimes satisfying the condition $\operatorname{div} C = 0$. Next we consider conformally flat $(WCZS)_4$ spacetimes. Finally, we characterise dust fluid and viscous fluid $(WCZS)_4$ spacetimes.

1. Introduction

The present paper is concerned with certain investigations in general relativity by the coordinate free method of differential geometry. In this method of study spacetime of general relativity is regarded as a connected four dimensional semi-Riemannian manifold (M^4, g) with Lorentzian metric g with signature $(-,+,+,+)$. The geometry of the Lorentzian manifold begins with the study of causal character of vectors of the manifold. It is due to this causality that the Lorentzian manifold becomes a convenient choice for the study of general relativity.

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* Corresponding author.

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The Einstein equation [22] implies that the energy-momentum tensor is of vanishing divergence. This requirement is satisfied if the energy momentum tensor is covariant constant. Chaki and Roy [6] showed that a general relativistic spacetime with covariant constant energy-momentum tensor is Ricci symmetric, that is, $\nabla S = 0$, where S is the Ricci tensor of the spacetime and ∇ denotes the covariant differentiation with respect to the metric tensor g. If however, $\nabla S \neq 0$, then such a spacetime may be called weakly Ricci symmetric [32]. We may say that the Ricci symmetric condition is only a special case of weakly Ricci symmetric manifold. Recently, Mantica and Molinari [14] introduced weakly Z symmetric manifolds which generalize the notion of weakly Ricci symmetric manifolds. Also De, Mantica and Suh [11] introduced the notion of weakly cyclic Z symmetric manifolds $(WCZS)_n$. As a special case Mantica and Suh [18] studied pseudo Z symmetric spacetimes. It is therefore meaningful to study the properties of weakly cyclic Z symmetric spacetimes in general relativity.

A non-flat Riemannian or a semi-Riemannian manifold $(M^n, g)(n > 2)$ is called weakly Ricci symmetric [32] if the Ricci tensor S is non-zero and satisfies the condition

$$(\nabla_X S)(U,V) = A(X)S(U,V) + B(U)S(V,X) + D(V)S(X,U), \tag{1.1}$$

where ∇ denotes the Levi-Civita connection and A, B and D are 1-forms which are non-zero simultaneously. Such an n-dimensional Riemannian manifold is denoted by $(WRS)_n$. If $A = B = D = 0$, then the manifold reduces to a Ricci symmetric manifold. The weakly Ricci symmetric spacetimes have been studied by De and Ghosh [9]. Among others it is proved that if in a weakly Ricci symmetric spacetime of non-zero scalar curvature the matter distribution is perfect fluid, then the acceleration vector and the expansion scalar are zero and such a spacetime can not admit heat flux. Several authors studied spacetimes in several ways such as conformally flat almost pseudo Ricci symmetric spacetimes by De, Özgür and De [10], m-projectively flat spacetimes by Zengin [35], pseudo Z symmetric spacetimes by Mantica and Suh [18] and many others.

According to Yano [34] a vector field V is torse-forming if

$$\nabla_X V = fX + \omega(X)V,$$

where f is a scalar function and ω is a 1-form. Its properties in pseudo-Riemannian manifolds were studied by Mikeš and Rachunek [20]. The vector is called concircular if ω is closed.

In a Riemannian or a semi-Riemannian manifold (M^n, g), $(n > 2)$, a $(0, 2)$ symmetric tensor is a generalized Z tensor if

$$Z(X,Y) = S(X,Y) + \phi g(X,Y), \tag{1.2}$$

where ϕ is an arbitrary scalar function. The scalar Z is obtained by contracting (1.2) over X and Y as follows:

$$Z = r + n\phi, \tag{1.3}$$

where the scalar curvature $r = \sum_{i=1}^{n} \varepsilon_i S(e_i, e_i)$, $g(e_i, e_i) = \varepsilon_i$, $\varepsilon_i = \pm 1$ and $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold. In a recent paper [14] the authors introduced weakly Z symmetric manifolds which is denoted by $(WZS)_n$. A Riemannian or a semi-Riemannian manifold is said to be weakly Z symmetric, denoted by $(WZS)_n$, if the generalized Z tensor satisfies the condition

$$(\nabla_X Z)(U, V) = A(X)Z(U, V) + B(U)Z(X, V) + D(V)Z(U, X), \tag{1.4}$$

where A, B and D are 1-forms not simultaneously zero. If $\phi = 0$, we recover from (1.4) a $(WRS)_n$, and as a particular case pseudo Ricci symmetric manifolds $(PRS)_n$ [4]. If $\phi = -\frac{r}{n}$ (classical Z tensor) and A is replaced by $2A$ and B and D are replaced by A, then

$$Z(U, V) = \frac{n-1}{n} P(U, V),$$

where $P(U, V)$ is the projective Ricci tensor considered by Chaki and Saha [7] and obtained by a contraction of the projective curvature tensor [12]. It is a generalization of the weakly Ricci symmetric manifolds [4], and pseudo Ricci symmetric manifolds [4] and pseudo projective Ricci symmetric manifolds [7].

A non-flat Riemannian or a semi-Riemannian manifold $(M^n, g)(n > 2)$ is called weakly cyclic Z symmetric [11] and denoted by $(WCZS)_n$, if the generalized Z tensor is non-zero and satisfies the condition

$$\begin{aligned} &(\nabla_X Z)(U, V) + (\nabla_U Z)(V, X) + (\nabla_V Z)(X, U) \\ &= A(X)Z(U, V) + B(U)Z(V, X) + D(V)Z(X, U), \end{aligned} \tag{1.5}$$

where Z is the generalized Z tensor. Such a manifold is denoted by $(WCZS)_n$. The classical Z tensor is obtained with the choice $\phi = -\frac{1}{n} r$, where r is the scalar curvature. Hereafter we refer to the generalized Z tensor simply as the Z tensor.

Recently two of the present authors studied pseudo Z symmetric Riemannian manifolds [16] and recurrent Z forms on Riemannian manifolds [17], that is, Riemannian manifolds on which the form $\Lambda_{(Z)l} = Z_{kl}\, dx^k$ satisfies the condition $D\Lambda_{(Z)l} = \beta \wedge \Lambda_{(Z)l}$, D being the exterior covariant derivative and $\beta = \beta_i\, dx^i$, the associated one-form. It should be noted that the concept of Z recurrent form embraces both pseudo Z symmetric and weakly Z symmetric manifolds.

On the other hand, Lorentzian manifolds with Ricci tensor S of the form

$$S(X,Y)=ag(X,Y)+bA(X)A(Y) \tag{1.6}$$

are often named perfect fluid spacetimes where a and b are scalars and the vector field ρ metrically equivalent to the 1-form A, that is, $g(X,\rho)=A(X)$ for all X, is a unit time like vector field, that is, $g(\rho,\rho)=-1$. It is well known that any Robertson–Walker spacetime is a perfect fluid spacetime [22]. The form (1.6) of the Ricci tensor is implied by Einstein's equation if the energy-matter content of the spacetime is a perfect fluid with velocity vector ρ. The scalars a and b are linearly related to the pressure p and the energy density σ measured in the locally comoving initial frame.

Geometers identify the special form (1.6) of the Ricci tensor as the defining property of quasi Einstein manifolds [5]. Semi-Riemannian quasi Einstein manifolds arose in the study of exact solutions of Einstein's equations. Robertson–Walker spacetimes are quasi Einstein [3]. The importance of the study of the quasi Einstein spacetime lies in the fact that this spacetime represents the present state of the universe, when the effects of viscosity and the heat flux have become negligible and the matter content of the universe may be considered as a perfect fluid.

Shepley and Taub [29] studied perfect fluid spacetime with equation of state $p=p(\sigma)$ and the additional condition $\operatorname{div} C=0$, where C is the conformal curvature tensor. A related result was obtained by Sharma [28]. De et al. [10] proved that conformally flat almost pseudo-Ricci symmetric spacetimes, that is,

$$(\nabla_X S)(Y,U)=\big(A(X)+B(X)\big)\,S(Y,U)+A(Y)S(X,U)+A(Z)S(X,Y),$$

are Robertson–Walker spacetimes.

Motivated by the above works, in the present paper we study $(WCZS)_4$ spacetimes. Study of such a spacetime partly deals with the physical structure of the universe at a large scale and describes physical processes occurring throughout its evolution and having observable consequences in the present time.

The paper is organized as follows: After introduction in Section 2, we prove that a $(WCZS)_4$ spacetime is a quasi Einstein spacetime. Section 3 is devoted to the study of $(WCZS)_4$ spacetimes satisfying the condition $\operatorname{div} C=0$. In this section we first show that such a spacetime satisfying the condition $\operatorname{div} C=0$ under certain assumption, the integral curves of the vector field ρ are geodesic and the vector field ρ is irrotational. Next we prove that such a spacetime is locally a product space. Also, we show that a $(WCZS)_4$ spacetime under certain condition is the Robertson–Walker spacetime. Section 4 deals with conformally flat $(WCZS)_4$ spacetimes. Finally, we study dust fluid and viscous fluid $(WCZS)_4$ spacetimes. Here we prove an interesting result which states that under certain condition a dust fluid $(WCZS)_4$ spacetime satisfying Einstein's field equation with cosmological constant is devoid of matter.

2. $(WCZS)_4$ spacetimes

PROPOSITION 2.1. *A $(WCZS)_4$ spacetime is a quasi Einstein spacetime.*

PROOF. Interchanging U and V in (1.5) we obtain

$$(\nabla_X Z)(V,U) + (\nabla_V Z)(U,X) + (\nabla_U Z)(X,V) = A(X)Z(V,U) + B(V)Z(U,X) + D(U)Z(X,V). \tag{2.1}$$

Subtracting (2.1) from (1.5) we get

$$\big[B(U) - D(U)\big]\, Z(V,X) + \big[D(V) - B(V)\big]\, Z(X,U) = 0.$$

which implies

$$\big[B(U) - D(U)\big]\, Z(V,X) = \big[B(V) - D(V)\big]\, Z(X,U), \tag{2.2}$$

where the symmetric properties of Z have been used. Suppose

$$E(X) = g(X,\rho) = B(X) - D(X), \tag{2.3}$$

for all vector fields X, where ρ is a unit time-like vector field associated with the 1-form E. Then the above relation reduces to

$$E(U)Z(V,X) = E(V)Z(X,U). \tag{2.4}$$

Taking a frame field and contracting (2.4) over X and V, we get

$$E(U)[r + 4\phi] = S(U,\rho) + \phi E(U).$$

which implies

$$S(U,\rho) = [r + 3\phi]E(U). \tag{2.5}$$

Putting $V = \rho$ in (2.4) yields

$$E(U)Z(\rho,X) = -Z(X,U). \tag{2.6}$$

Using (1.2) in (2.6) we obtain

$$E(U)\big[\,S(\rho,X) + \phi g(X,\rho)\big] = -\big[\,S(X,U) + \phi g(X,U)\big]\,. \tag{2.7}$$

Using (2.5) in (2.7) we have

$$E(U)\big[\{r+3\phi\}E(X)+\phi E(X)\big] = -\big[S(X,U)+\phi g(X,U)\big].$$

which implies

$$S(X,U) = -\phi g(X,U) - (r+4\phi)E(X)E(U),$$

that is,

$$S(X,U) = ag(X,U) + bE(X)E(U), \tag{2.8}$$

where $a=-\phi$, $b=-(r+4\phi)$. $\square$

3. $(WCZS)_4$ spacetime satisfying the condition $\operatorname{div} C = 0$

Suppose (M^n,g) is a semi-Riemannian manifold of dimension n and X is any vector field on M. Then the divergence of the vector field X, denoted by $\operatorname{div} X$, is defined as

$$\operatorname{div} X = \sum_{i=1}^{n} \varepsilon_i g(\nabla_{e_i} X, e_i),$$

where $\{e_i\}$ is an orthonormal basis of the tangent space T_pM at any point $p \in M$ and $\varepsilon_i = \pm$. Again, if K is a tensor field of type (1,3), then its divergence $\operatorname{div} K$ is a tensor field of type (0,3) defined as

$$(\operatorname{div} K)(X_1,\dots,X_3) = \sum_{i=1}^{n} \varepsilon_i g\big((\nabla_{e_i} K)(X_1,\dots,X_3), e_i\big).$$

In this section we assume that the $(WCZS)_4$ spacetimes satisfy the condition $\operatorname{div} C = 0$, where C denotes the Weyl conformal curvature tensor and "div" denotes divergence. Hence we have [12]

$$(\nabla_X S)(Y,U) - (\nabla_U S)(Y,X) = \frac{1}{6}\big[g(Y,U)dr(X) - g(X,Y)dr(U)\big]. \tag{3.1}$$

Using (2.8) in (3.1) we have

$$\begin{aligned} &da(X)g(Y,U) + db(X)E(Y)E(U) + b\big[(\nabla_X E)(Y)E(U) \\ &+ (\nabla_X E)(U)E(Y)\big] - da(U)g(Y,X) - db(U)E(Y)E(X) \\ &- b\big[(\nabla_U E)(Y)E(X) + (\nabla_U E)(X)E(Y)\big] \end{aligned} \tag{3.2}$$

$$= \frac{1}{6}\left[g(Y,U)dr(X) - g(X,Y)dr(U)\right].$$

Taking a frame field and contracting X and Y we get

$$-3da(U) + db(\rho)E(U) + bE(U)(\delta E) + b(\nabla_\rho E)(U) + db(U) = -\frac{1}{2}dr(U), \tag{3.3}$$

where

$$\delta E = \sum_{i=1}^{n} \varepsilon_i (\nabla_{e_i} E)(e_i).$$

Putting $X = Y = \rho$ in (3.2) yields

$$b(\nabla_\rho E)(U) = da(\rho)E)(U) - db(\rho)E(U) + da(U) - db(U) - \frac{1}{6}\left[dr(\rho)E(U) + dr(U)\right]. \tag{3.4}$$

Substituting (3.4) in (3.3) we get

$$-2da(U) + da(\rho)E(U) + bE(U)(\delta E) - \frac{1}{6}dr(\rho)E(U) = -\frac{1}{3}dr(U). \tag{3.5}$$

Putting $U = \rho$ in (3.5) we obtain

$$-3da(\rho) - b(\delta E) = -\frac{1}{2}dr(\rho). \tag{3.6}$$

Using (3.6) in (3.5) we get

$$-2da(U) - 2da(\rho)E(U) + \frac{1}{3}dr(\rho)E(U) = -\frac{1}{3}dr(U). \tag{3.7}$$

Let $r = a$, then

$$dr(U) = da(U). \tag{3.8}$$

and

$$db(U) = 3da(U). \tag{3.9}$$

Putting (3.8) in (3.7), yields

$$da(U) = -da(\rho)E(U). \tag{3.10}$$

Using (3.8) in (3.10) we get

$$dr(U) = -dr(\rho)E(U). \tag{3.11}$$

Putting $Y = \rho$ in (3.2) and using (3.10) we have

$$(\nabla_U E)(X) - (\nabla_X E)(U) = 0, \tag{3.12}$$

since $b \neq 0$. This means that the 1-form E defined by (2.3) is closed, that is,

$$dE(X, Y) = 0.$$

Hence it follows that

$$g(\nabla_X \rho, Y) = g(\nabla_Y \rho, X) \tag{3.13}$$

for all X, Y.

Now using $Y = \rho$ in (3.13) we get

$$g(\nabla_X \rho, \rho) = g(\nabla_\rho \rho, X). \tag{3.14}$$

Since $g(\nabla_X \rho, \rho) = 0$, from (3.14) it follows that $g(\nabla_\rho \rho, X) = 0$ for all X. Hence $\nabla_\rho \rho = 0$. This means that the integral curves of the vector field ρ are geodesic and ρ is irrotational. Therefore we can state the following:

THEOREM 3.1. *In a* $(WCZS)_4$ *spacetime satisfying the condition* $\operatorname{div} C = 0$ *under the assumption* $r = a$, *the integral curves of the vector field* ρ *are geodesic and the vector field* ρ *is irrotational.*

Using (3.10) and (3.11) in (3.4) we obtain

$$(\nabla_\rho E)(U) = 0, \tag{3.15}$$

since $b \neq 0$. Now we consider the scalar function

$$f = \frac{1}{6}\,\frac{dr(\rho)}{b}. \tag{3.16}$$

Then using (3.9) we get

$$\nabla_X f = \frac{1}{2}\,\frac{dr(\rho)}{b^2} dr(X) + \frac{1}{6b} d^2 r(\rho, X). \tag{3.17}$$

On the other hand, (3.11) implies

$$d^2 r(Y, X) = -d^2 r(\rho, Y)E(X) - dr(\rho)(\nabla_Y E)(X).$$

from which we get

$$d^2 r(\rho, Y)E(X) = d^2 r(\rho, X)E(Y), \tag{3.18}$$

since $(\nabla_X E)(Y) = (\nabla_Y E)(X)$ and $d^2r(Y,X) = d^2r(X,Y)$.

Putting $X = \rho$ in (3.18), it follows that

$$d^2r(Y,\rho) = -d^2r(\rho,\rho)E(Y). \tag{3.19}$$

Then using (3.19) in (3.17) we obtain

$$\nabla_X f = -\frac{dr(\rho)}{2b^2} dr(\rho)E(X) - \frac{1}{6b} d^2r(\rho,\rho)E(X).$$

which implies that

$$\nabla_X f = \mu E(X), \tag{3.20}$$

where $\mu = \frac{1}{6b}\big[- d^2r(\rho,\rho) - 3dr(\rho)dr(\rho)\big]$.

Using (3.20), it is easy to show that

$$\omega(X) = \frac{1}{6}\frac{dr(\rho)}{b}E(X) = fE(X)$$

is closed. In fact, $d\omega(X,Y) = 0$.

Using (3.10), (3.11), (3.12) in (3.2) we have

$$\begin{aligned} & -dr(\rho)E(X)g(Y,U) + b\big[(\nabla_X E)(Y)E(U) + (\nabla_X E)(U)E(Y)\big] \\ & + dr(\rho)E(U)g(Y,X) - b\big[(\nabla_U E)(Y)E(X) + (\nabla_U E)(X)E(Y)\big] \\ & = \frac{1}{6}\big[- g(Y,U)dr(\rho)E(X) + g(X,Y)dr(\rho)E(U)\big]. \end{aligned} \tag{3.21}$$

Putting $U = \rho$ in (3.21) and using (3.15) we obtain

$$(\nabla_X E)(Y) = \left(f - \frac{dr(\rho)}{b}\right)g(X,Y) + \left(\omega(X) - \frac{dr(\rho)}{b}E(X)\right)E(Y). \tag{3.22}$$

From (3.22) it follows that

$$\nabla_X \rho = \left(f - \frac{dr(\rho)}{b}\right)X + \left(\omega(X) - \frac{dr(\rho)}{b}E(X)\right)\rho. \tag{3.23}$$

Let $\rho^\perp$ denote the 3-dimensional distribution in a $(WCZS)_4$ spacetime orthogonal to ρ. If X and Y belong to $\rho^\perp$, then

$$g(X,\rho) = 0. \tag{3.24}$$

and

$$g(Y,\rho) = 0. \tag{3.25}$$

Since $(\nabla_X g)(Y,\rho)=0$, it follows from (3.23) and (3.25) that

$$g(\nabla_X Y,\rho)=g(\nabla_X \rho, Y)=\left(f-\frac{dr(\rho)}{b}\right)g(X,Y).$$

Similarly, we get

$$g(\nabla_Y X,\rho)=g(\nabla_Y \rho, X)=\left(f-\frac{dr(\rho)}{b}\right)g(X,Y).$$

Hence

$$g(\nabla_X Y,\rho)=g(\nabla_Y X,\rho). \tag{3.26}$$

Now $[X,Y]=\nabla_X Y-\nabla_Y X$ and therefore by (3.26) we obtain

$$g\big([X,Y],\rho\big)=g(\nabla_X Y-\nabla_Y X,\rho)=0.$$

Hence $[X,Y]$ is orthogonal to ρ. That is $[X,Y]$ belongs to $\rho^{\perp}$. Thus the distribution $\rho^{\perp}$ is involutive [8]. Hence from Frobenius' theorem [8] it follows that $\rho^{\perp}$ is integrable. This implies that if a $(WCZS)_4$ spacetime satisfies $\operatorname{div} C=0$, then it is locally a product space. Hence we have the following:

THEOREM 3.2. *If a $(WCZS)_4$ spacetime satisfies $\operatorname{div} C=0$ and fulfilling the condition $r=a$, then it is locally a product space.*

From (3.22) one can write

$$(\nabla_X E)(Y)=\beta g(X,Y)+\gamma(X)E(Y), \tag{3.27}$$

where

$$\beta=\left(f-\frac{dr(\rho)}{b}\right)\quad\text{and}\quad\gamma(X)=\left(\omega(X)-\frac{dr(\rho)}{b}E(X)\right).$$

Obviously γ is closed. In local components this reads $\nabla_k E_j=\gamma_k E_j+\beta g_{kj}$.

Therefore the vector field ρ corresponding to the 1-form E defined by $g(X,\rho)=E(X)$ is a concircular vector field [27,33]. Hence we can state the following:

PROPOSITION 3.1. *If a $(WCZS)_4$ spacetime satisfies $\operatorname{div} C=0$ and fulfills the condition $r=a$, then ρ is a concircular vector field.*

Yano [34] proved that in order that a Riemannian space admits a concircular vector field, it is necessary and sufficient that there exists a coordinate system with respect to which the fundamental quadratic differential form may be written in the form

$$ds^2=\left(dx^1\right)^2+q(t)^2 g^*_{\alpha\beta}\,dx^\alpha\,dx^\beta, \tag{3.28}$$

where $g^*_{\alpha\beta} = g^*_{\alpha\beta}(x^\gamma)$ are functions of x^γ only $(\alpha, \beta, \gamma = 2, 3, \dots, n)$ and $q = q(x^1) \neq \text{const.}$ is a function of x^1 only.

Now if E_j is closed, it is locally a gradient of a suitable scalar function, that is, $E_j = \nabla_j \sigma$ (see [24] pp. 242–243); setting $X_j = E_j e^{-\sigma}$ we have (see [15] and [19])

$$\nabla_k X_j = e^{-\sigma}(\nabla_k E_j - E_j \nabla_k \sigma)$$

$$= e^{-\sigma}\big[(\nabla_k \sigma)E_j + \beta g_{kj} - E_j(\nabla_k \sigma)\big] = (e^{-\sigma}\beta) g_{kj}$$

and consequently $\nabla_k X_j = \theta g_{kj}$, being $\theta = e^{-\sigma}\beta$ a scalar function and $X_j X^j = -e^{-2\sigma} < 0$ a time-like vector. The previous equation can be written in the form $\nabla_k X_j + \nabla_j X_k = 2\theta g_{kj}$, that is, X_j is a conformal Killing vector [31]. We recall now the definition of a generalized Robertson–Walker spacetime [1,25,26]

Definition 3.1. An $n \geqq 3$-dimensional Lorentzian manifold is a generalized Robertson–Walker spacetime if the metric takes the local shape

$$ds^2 = -(dt)^2 + q(t)^2 g^*_{\alpha\beta}\, dx^\alpha\, dx^\beta, \tag{3.29}$$

where $g^*_{\alpha\beta} = g^*_{\alpha\beta}(x^\gamma)$ are functions of x^γ only $(\alpha, \beta, \gamma = 2, 3, \dots, n)$ and q is a function of t only.

The generalized Robertson–Walker spacetime is thus the warped product $-1 \times q^2 M^*$ [1,25,26] where M^* is an $n-1$ dimensional Riemannian manifold. If M^* is a 3-dimensional Riemannian manifold of constant curvature, the spacetime is called Robertson–Walker spacetime. The following deep result was recently proved in [2].

Theorem [2]. *Let M be an $n \geqq 2$ dimensional Lorentzian manifold. Then the spacetime is a generalized Robertson–Walker spacetime if and only if it admits a time-like vector of the form $\nabla_k X_j = \theta g_{kj}$.*

In view of these results, if a $(WCZS)_4$ spacetime satisfies $\operatorname{div} C = 0$ and fulfills the condition $r = a$, then it admits a concircular vector field rescalable to a time-like vector of the form $\nabla_k X_j = \theta g_{kj}$ and so becomes a generalized Robertson–Walker spacetime, that is, it is the warped product $-1 \times q^2 M^*$ where M^* is a 3-dimensional Riemannian manifold. Gębarowski [23] proved that the warped product $-1 \times q^2 M^*$ satisfies $\operatorname{div} C = 0$ if and only if M^* is Einstein. But a 3-dimensional Einstein manifold is a manifold of constant curvature. Hence we conclude that

Theorem 3.3. *If a $(WCZS)_4$ spacetime satisfies $\operatorname{div} C = 0$ and fulfills the condition $r = a$, then the spacetime is the Robertson–Walker spacetime.*

4. Conformally flat $(WCZS)_4$ spacetimes

This section is devoted to the study of conformally flat $(WCZS)_4$ spacetimes. In a conformally flat 4-dimensional Lorentzian manifold the curvature tensor R is of the form

$$R(X,Y)U = \frac{1}{2}\big[S(Y,U)X - S(X,U)Y + g(Y,U)QX - g(X,U)QY\big] - \frac{r}{6}\big[g(Y,U)X - g(X,U)Y\big], \tag{4.1}$$

where Q is the Ricci operator defined by $g(QX,Y) = S(X,Y)$.

Using (2.8) in (4.1) yields

$$\begin{aligned} R(X,Y)U = \frac{1}{2}\big[&ag(Y,U) + bE(Y)E(U) - ag(X,U)Y - bE(X)E(U)Y \\ &+ ag(Y,U)X + bg(Y,U)E(X)\rho - ag(X,U)Y - bg(X,U)E(Y)\rho\big] \\ &-\frac{r}{6}\big[g(Y,U)X - g(X,U)Y\big] \end{aligned}$$

Let $\rho^{\perp}$ denote the 3-dimensional distribution in a conformally flat $(WCZS)_4$ spacetimes orthogonal to ρ, then

$$R(X,Y)U = \left(a - \frac{r}{6}\right)\big[g(Y,U)X - g(X,U)Y\big] \tag{4.2}$$

for all $X, Y \in \rho^{\perp}$ and

$$R(X,\rho)\rho = -\left(a - \frac{r}{6}\right)X, \tag{4.3}$$

for every $X \in \rho^{\perp}$. According to Karchar [13], a Lorentzian manifold is called infinitesimal spatially isotropic relative to a timelike unit vector field ρ if its curvature tensor R satisfies the relations

$$R(X,Y)U = l\big[g(Y,U)X - g(X,U)Y\big]$$

for all $X, Y, U \in \rho^{\perp}$ and $R(X,\rho)\rho = mX$ for all $X \in \rho^{\perp}$, where l, m are real valued functions on the manifold. So by virtue of (4.2) and (4.3) we can state the following:

THEOREM 4.1. *A conformally flat $(WCZS)_4$ spacetime is infinitesimally spatially isotropic relative to the unit timelike vector field ρ.*

5. Dust fluid and viscous fluid $(WCZS)_4$ spacetimes

In a dust or pressureless fluid spacetime, the energy momentum tensor T is of the form [30]

$$T(X,Y) = \sigma E(X)E(Y), \tag{5.1}$$

where σ is the energy density of the dust-like matter and E is a non-zero 1-form such that $g(X,\rho) = E(X)$, for all X, ρ being the velocity vector field of the flow, that is, $g(\rho,\rho) = -1$. In Proposition 2.1, it is proved that a $(WCZS)_4$ spacetime is a quasi Einstein spacetime, that is,

$$S(X,Y) = ag(X,Y) + bE(X)E(Y), \tag{5.2}$$

where $a = -\phi$, $b = -(r + 4\phi)$. Einstein's field equation with cosmological constant is

$$S(X,Y) - \frac{r}{2}g(X,Y) + \lambda g(X,Y) = \kappa T(X,Y), \tag{5.3}$$

where λ is the cosmological constant and κ is the gravitational constant.

Using (5.1) and (5.2) in (5.3), we obtain

$$\left(a - \frac{r}{2} + \lambda\right) g(X,Y) + bE(X)E(Y) = \kappa\sigma E(X)E(Y). \tag{5.4}$$

Taking a frame field after contraction over X and Y we have

$$4\left(a - \frac{r}{2} + \lambda\right) - b = -\kappa\sigma$$

which implies

$$\lambda = \frac{1}{4}(2r - 4a + b - \kappa\sigma). \tag{5.5}$$

Again, if we put $X = Y = \rho$ in (5.4), we get

$$-\left(a - \frac{r}{2} + \lambda\right) + b = \kappa\sigma$$

which implies that

$$\lambda = \frac{r}{2} - a + b - \kappa\sigma. \tag{5.6}$$

Combining equation (5.5) and (5.6), we obtain

$$\frac{\sigma}{\kappa} = -\frac{(r + 4\phi)}{\kappa}.$$

Therefore $\sigma = -\frac{Z}{\kappa}$, using (1.3). Thus we can state the following:

THEOREM 5.1. *A dust fluid* $(WCZS)_4$ *spacetime satisfying Einstein's field equation with cosmological constant is vacuum, provided the scalar Z vanishes.*

Let us consider the energy momentum tensor T of a viscous fluid spacetime in the following form [21,22]:

$$T(X,Y) = pg(X,Y) + (\sigma + p)E(X)E(Y) + P(X,Y), \tag{5.7}$$

where σ, p are the energy density and isotropic pressure respectively and P denotes the anisotropic pressure of the fluid.

Using (5.2) and (5.3) in (5.7), we get

$$\begin{aligned} &\left(a - \frac{r}{2} + \lambda\right) g(X,Y) + bE(X)E(Y) \\ &= \kappa\big[pg(X,Y) + (\sigma + p)E(X)E(Y) + P(X,Y)\big]. \end{aligned} \tag{5.8}$$

Putting $X = Y = \rho$ in (5.8), yields

$$-\left(a - \frac{r}{2} + \lambda\right) + b = \kappa\big[-p + (\sigma + p) + I\big],$$

where $I = P(\rho,\rho)$, which implies

$$\sigma = -\frac{1}{\kappa}\left[\frac{r}{2} + \lambda + 3\phi + I\kappa\right]. \tag{5.9}$$

Again contracting (5.8) over X and Y, we get

$$4\left(a - \frac{r}{2} + \lambda\right) - b = \kappa\big[4p - (\sigma + p) + J\big],$$

where $J =$Trace of P, which implies

$$p = \frac{1}{\kappa}\left[\lambda - \frac{r}{2} - \phi - \frac{\kappa(I + J)}{3}\right] \tag{5.10}$$

Thus we can state the following:

THEOREM 5.2. *In a viscous fluid* $(WCZS)_4$ *spacetime obeying Einstein's equation with cosmological constant, the energy density and the isotropic pressure are given by the relations* (5.9) *and* (5.10).

We now discuss whether a viscous fluid $(WCZS)_4$ spacetime can admit heat flux or not. Let the energy momentum tensor T be of the following form [21,22]:

$$T(X,Y) = pg(X,Y) + (\sigma + p)E(X)E(Y) + E(X)F(Y) + E(Y)F(X), \tag{5.11}$$

where $F(X) = g(X, \xi)$ for all vector fields X; ξ being the heat flux vector field. Thus we have $g(\rho, \xi) = 0$, that is, $F(\rho) = 0$.

Using (5.2) and (5.3) in (5.11) we obtain

$$\left(a - \frac{r}{2} + \lambda\right) g(X,Y) + bE(X)E(Y) \tag{5.12}$$

$$= \kappa\big[pg(X,Y) + (\sigma + p)E(X)E(Y) + E(X)F(Y) + E(Y)F(X)\big].$$

Putting $Y = \rho$ in (5.12), yields

$$\left(a - \frac{r}{2} + \lambda - b + \sigma\kappa\right) E(X) + \kappa F(X) = 0,$$

which implies

$$F(X) = -\frac{1}{\kappa}\left(3\phi + \frac{r}{2} + \lambda + \kappa\sigma\right) E(X), \tag{5.13}$$

where $a = -\phi$, and $b = -(r + 4\phi)$.

Thus we can state the following:

THEOREM 5.3. *A viscous fluid $(WCZS)_4$ spacetime obeying Einstein's field equation with cosmological constant admits heat flux, provided $3\phi + \frac{r}{2} + \lambda + \kappa\sigma \neq 0$.*

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